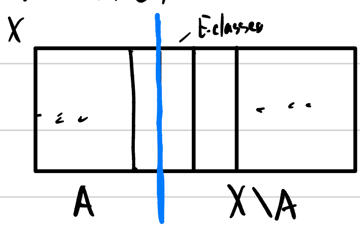
Ergodic Theory and Measured Group Theory
Lecture 3

Ergodicity. "Ergodic" is a made-up word, if it was me, I would've used "atomic" instead. Ergodicity is defined not just for a transformation or group action on (X, μ), but more generally, for an equivalence relation on (X, μ). For a meas. trans. $T: (X,\mu) \to (X,\mu)$, having the orbit eq. rel. E_T in mind, we define:

Def. Let E be an equiv. relation on a measure space (X, μ). E is called *ergodic* if every measurable E-invariant set $A \subseteq X$ is null or conull ($:\Leftrightarrow X \setminus A$ is null).



For a meas. transformation $T: (X,\mu) \to (X,\mu)$, T is ergodic $:\Leftrightarrow E_T$ is ergodic, i.e. X doesn't partition into two positively-measured T-invariant sets.

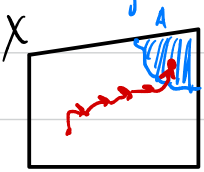
Equivalent definitions. For a meas. transformation $T: (X,\mu) \to (X,\mu)$, TFAE:

(1) T is ergodic.

(2) Functional def. Every T-invariant meas. function $f: X \to Y$ (where Y

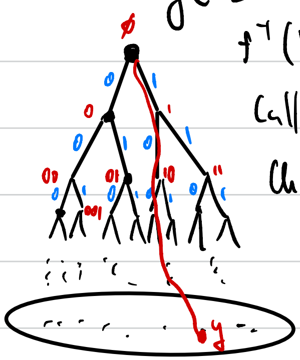
is any standard Borel space) is constant a.e.

(3) **Density def.** For each positively-measured set A, a.e. orbit intersects it, i.e. a.e. $x \in X$ has $[x]_{E_T} \cap A \neq \emptyset$. Equivalently, $[A]_{E_T}$ is conull.



Proof. (2) $\Rightarrow$ (1). For any T-inv. meas. set $A \subseteq X$, take $f := \mathbb{1}_A$. Then $f : X \to \{0,1\}$ is T-invariant, hence $f \equiv 1$ a.e. or $f \equiv 0$ a.e.

(1) $\Rightarrow$ (2). Proof for $Y := 2^{\mathbb{N}}$. Let $f : X \to 2^{\mathbb{N}}$ be a T-inv. meas. func. Note that by the invariance of f, $f^{-1}(B)$ is T-invariant for every Borel set $B \subseteq 2^{\mathbb{N}}$. We want to show that $\exists\, y \in 2^{\mathbb{N}}$ s.t. $f^{-1}(y)$ is a conull $\subseteq X$.



$f^{-1}(V_\emptyset)$ is X so it's conull, where $V_s := \{x \in 2^{\mathbb{N}} : x \supseteq s\}$.
Call a vertex $s \in 2^{<\mathbb{N}}$ heavy if $f^{-1}(V_s)$ is conull.
Chasing heavy vertices, we build an infinite branch $y = (y_n)$, i.e. $f^{-1}(V_{y|n})$ is conull. $\bigcap_{n \in \mathbb{N}} f^{-1}(V_{y|n})$ is still conull, but

$$\bigcap_{n \in \mathbb{N}} f^{-1}(V_{y|n}) = f^{-1}(y).$$

For any st. Borel Y, (or just use the isom. thm $Y \overset{\text{Borel}}{\cong} 2^{\mathbb{N}}$) take a Polish top. on it and let $\mathcal{V}$ be a ctbl basis for that top. $f^{-1}(Y)$ is conull, so in some complete metric

$\exists\, V_1 \in \mathcal{V}$ of diameter $\frac{1}{2}$ s.t. $f^{-1}(V_1)$ is comull. Then $\exists\, \overline{V}_2 \subseteq V_1$ and $V_2 \in \mathcal{V}$ of diam. $\frac{1}{3}$ s.t. $f^{-1}(V_2)$ is comull. Then by completeness $\bigcap_n \overline{V}_n \neq \emptyset$ and has exactly one point y, so $f^{-1}(y)$ is comull.

$$\bigcap_n \overline{V}_n = \bigcap_n V_n.$$

(3) ⇒ (1). Suppose that A is T-inv. and not null, then $[A]_{E_T} = A$, (3) gives that A is conull.

(1) ⇒ (3). If A has positive measure, then $[A]_{E_T}$ also has positive measure and is T-invariant. By ergodicity, $[A]_{E_T}$ is conull <u>if we knew that $[A]_{E_T}$ is measurable.</u>

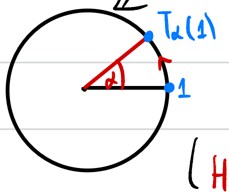
<u>Remark.</u> We're using here that $[A]_{E_T}$ is measurable. This is true because of a deep theorem from descriptive set theory, but we can avoid using this as follows: if A has positive measure, so does A' from the Poincaré recurrence lemma, i.e. A' is T-forward-recurrent and $A' =_\mu A$. But then $[A']_{E_T}$ is measurable and still not null, so must be conull. □

Examples of ergodic transformations.

o Irrational rotation. Let $\alpha \in [-\pi, \pi)$ be s.t. $\frac{\alpha}{\pi} \in \mathbb{R} \setminus \mathbb{Q}$.

Let $\mathbb{T} \cong \mathbb{R}/\mathbb{Z} \cong S^1$ and let $T_\alpha : \mathbb{T} \to \mathbb{T}$ be the rotation by α,

i.e.

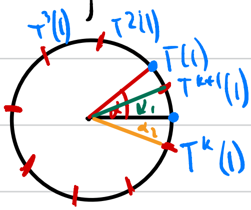


Note that if $\alpha/\pi \in \mathbb{Q}$ then T_α is periodic, i.e. every orbit is finite, hence nonergodic.

(HW. Why nonergodic?)

Lemma. If $\alpha/\pi \notin \mathbb{Q}$ then each T_α-orbit is dense in $\mathbb{T}$.

Proof. Enough to prove that the orbit of 1 is dense since every other orbit is just a rotation/translate of this.



$\exists k$ s.t. $T^k(1)$ is below 1 and $T^{k+1}(1)$ is above. Then $T_\alpha^{k+1} = T_{\alpha_1}$ and $T_\alpha^{-k} = T_{\alpha_2}$.

But one of α_1, α_2 is $\leq \frac{1}{2}\alpha$. So

Doing this enough times gives arbitrarily small angles. $\square$

99% Lemma (special case of Lebesgue diff. theorem). For any positively-measured subset $A \subseteq [0,1)$, there is an open interval $I \subseteq [0,1)$ s.t. $\geq 99\%$ of I is occupied by A, i.e. $\frac{\lambda(A \cap I)}{\lambda(I)} = 0.99$.